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Gauge covariant approximations in scalar electrodynamics

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Abstract. In electrodynamics the longitudinal components of all charged-particle Green functions are determined by the charged-particle propagator. Inserting these components into the Dyson–Schwinger equation leads to an integral equation for the propagator itself which may be solved and used as a basis for an iterative determination of the transverse Green functions' components. We find the solution for electrodynamics of scalar mesons in analogy with recent work on spinor electrodynamics.

1. Introduction

Scalar electrodynamics is in some respects simpler and in other respects more complicated than spinor electrodynamics. The simplicity stems from the fact that all the γ-matrix algebra is absent and therefore all Green functions are straightforward tensor functions of the invariants; the complications arise from the existence of quadrilinear vertices $e^2\phi^+\phi A^2$ which increase the multiplicity of graphs that need to be considered in any particular order of perturbation theory. One can, of course, get rid of the second difficulty by using Kemmer's β-formalism but the price to be paid is in the resulting β-algebra which can be quite difficult to handle for multi-meson amplitudes; therefore we shall avoid it in what follows—in any case it ought to be equivalent to the more usual formalism.

Being a gauge theory, electrodynamics is characterised by its particular Ward–Takahashi identities, identities such as

$$k^\lambda \Delta(p)\Gamma_\lambda(p, p-k)\Delta(p-k) = \Delta(p-k) - \Delta(p)$$

$$k'^\nu \Delta(p')\Gamma_{\nu\mu}(p'k', pk)\Delta(p) \qquad (1)$$

$$= \Delta(p+k)\Gamma_\mu(p+k, p)\Delta(p) - \Delta(p')\Gamma_\mu(p', p'-k)\Delta(p'-k)$$

where the Γ are the amputated, connected Green functions. These identities determine the purely longitudinal pieces of the Green functions in terms of the complete renormalised charged-meson propagator Δ. They thus supply *some* information about the structure of Γ, in spite of the fact that the transverse components remain unknown without further explicit evaluation. This information can be used to good effect for finding non-perturbative solutions of the field equations which are *gauge-covariant.* Having found these solutions, one can proceed step by step to compute the transverse components of Γ from the Dyson–Schwinger equations. This is the basis of Salam's gauge technique (1963). In an earlier paper (Delbourgo and West 1977), we applied the method to spinor electrodynamics and were able to determine the initial spinor propagator and zeroth-order Green functions for starting the iteration scheme. In this paper, we wish to exhibit the parallel calculations for scalar meson electrodynamics and the similarities with the spinor case. Non-Abelian gauge theories bear a greater resemblance to scalar than to spinor electrodynamics and it will be a non-trivial exercise to extend our work and cover this case in order to understand better how such non-perturbative solutions might possibly lead to quark confinement as suggested by Nash and Stuller (1976), and Pagels (1976, 1977).

2. Zeroth-order Green functions

The initial Green functions to be used as the basis of an iteration scheme are determined solely by Δ and are defined to reduce to the Born approximation when the charged meson line is on mass shell. In analogy with spinor electrodynamics, if we begin with the Lehmann spectral representation,

$$\Delta(p)=\int \frac{\rho(s)\,\mathrm{d}s}{p^2-s+\mathrm{i}0} \tag{2}$$

then the appropriate zeroth-order solutions of the Ward identities (1) are given by

$$\Delta^{(0)}(p)\Gamma^{(0)}_{\lambda}(p,p-k)\Delta^{(0)}(p-k)=\int \frac{(2p-k)_\lambda\rho(s)\,\mathrm{d}s}{(p^2-s)[(p-k)^2-s]} \tag{3}$$

$$\Delta^{(0)}(p')\Gamma^{(0)}_{\nu\mu}(p'k',pk)\Delta^{(0)}(p)$$
$$=\int \frac{\rho(s)\,\mathrm{d}s}{(p'^2-s)(p^2-s)}\left(2\eta_{\nu\mu}-\frac{(2p'+k')_\nu(2p+k)_\mu}{(p+k)^2-s}-\frac{(2p-k')_\nu(2p'-k)_\mu}{(p'-k)^2-s}\right). \tag{4}$$

In other words, they are spectrally weighted Born terms. The Γ satisfy the defining properties as one can easily check. It only remains to find the spectral function and this we do by means of the Dyson–Schwinger equation for the propagator (figure 1),

$$\Delta^{-1}(p)=Z_\phi(p^2-m_0^2)-\mathrm{i}e^2Z_\phi\int \Gamma_\nu(p,p-k)\Delta(p-k)D^{\mu\nu}(k)(2p-k)_\mu\,\bar{\mathrm{d}}^4k$$
$$+2e^4Z_\phi\int \Gamma_{\mu\nu}(p,-k;p',k')D^{\mu\lambda}(k)D^{\nu\lambda}(k')\,\bar{\mathrm{d}}^4k\,\bar{\mathrm{d}}^4k'\Delta(p')$$
$$+\text{photon tadpole term} \tag{5}$$

if we neglect photon dressing in zeroth order (which, in any case, has no effect on the gauge identities). Armed with $\Delta^{(0)}$, we can go on to compute the transverse components in $\Gamma^{(1)}$ and in $D^{(1)}_{\mu\nu}$ by iteration in the manner laid out previously (Delbourgo and Salam 1964, Delbourgo and West 1977), and so on to higher $\Gamma^{(n)}$. However, it is important to

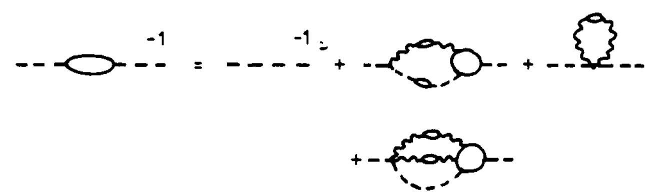

Figure 1. Dyson–Schwinger equation for the meson propagator in scalar electrodynamics.

realise that already in zeroth order we are dealing with a non-perturbative gauge-covariant approximation of the electrodynamics, one which encompasses previous (gauge-deficient) self-consistent methods.

The zeroth-order equation contained in (5) is

$$
\begin{aligned}
Z_\phi^{-1} = {} & (p^2 - m_0^2)\Delta(p) - \mathrm{i}e^2 \int \bar{\mathrm{d}}^4k \,\mathrm{d}s \frac{\rho(s)(2p-k)_\mu(2p-k)_\nu}{(p^2-s)[(p-k)^2-s]}\left(-\frac{\eta^{\mu\nu}}{k^2}+\frac{k^\mu k^\nu(1-a)}{k^4}\right) \\
& + 2e^4 \int \frac{\mathrm{d}s\,\rho(s)\,\bar{\mathrm{d}}^4k\,\bar{\mathrm{d}}^4k'}{(p^2-s)k^2k'^2}\left(\eta_{\mu\nu} - \frac{(2p-k)_\mu(2p-2k-k')_\nu}{(p-k)^2-s}\right)\frac{1}{(p-k-k')^2-s} \\
& \times[-\eta^{\lambda\mu} + k^\lambda k^\mu(1-a)/k^2](-\eta^{\lambda\nu} + k'^\lambda k'^\nu(1-a)/k'^2] \\
& + \text{photon tadpole term}
\end{aligned}
$$

upon adopting a convariant photon gauge specified by parameter a. More simply,

$$
Z_\phi^{-1} = \int \frac{\mathrm{d}s\,\rho(s)}{p^2-s}(p^2 - m_0^2 + \Sigma(p^2, s)) + \text{photon tadpole term} \tag{6}
$$

where $\Sigma(p^2, s)$ is given by the proper diagrams of figure 2 corresponding to an internal meson of mass $s^{1/2}$. Picking out the pole term by putting

$$
\rho(s) = \delta(s - m^2) + \sigma(s)
$$

Figure 2. Absorptive parts of Σ in the integral equation for the meson propagator.

and making the necessary renormalisations in (6), we are left with the integral equation

$$
\frac{\Sigma(p^2, m^2)}{p^2-m^2} + \int \frac{\mathrm{d}s\,\sigma(s)}{p^2-s+\mathrm{i}0}(s - m^2 + \Sigma(p^2, s)) = 0 \tag{7}
$$

where Σ is once-subtracted:

$$
\Sigma(p^2, m^2) = \frac{(p^2-m^2)}{\pi}\int \frac{\mathrm{Im}\,\Sigma(s, m^2)\,\mathrm{d}s}{(s-m^2)(s-p^2-\mathrm{i}0)}. \tag{8}
$$

Upon taking imaginary parts of (7), this then gives the linear equation

$$
(s-m^2)\sigma(s) = \frac{\mathrm{Im}\,\Sigma(s, m^2)}{\pi(s-m^2)} + \int \mathrm{d}s'\,\sigma(s')\frac{\mathrm{Im}\,\Sigma(s, s')}{\pi(s-s')} \tag{9}
$$

in complete analogy with the corresponding equation for the spinor case. The only difference is that now $\mathrm{Im}\,\Sigma$ receives e^2 *and* e^4 contributions.

We can find Σ by elementary calculation (the appendix gives relevant details). The order e^2 contribution is well known but the e^4 term is perhaps not:

$$
\begin{aligned}
\mathrm{Im}\,\Sigma(p^2, m^2) = {} & \frac{e^2(a-3)(p^4-m^4)}{16\pi p^2} - \frac{e^4}{64\pi^3p^2}\int_{m^2}^{p^2}\frac{\mathrm{d}t}{t}\{[1+a+\tfrac{5}{8}(1-a)^2](p^2-t)(m^2-t) \\
& + \tfrac{1}{8}[(2a-3)m^2-3t][(2a-3)p^2-3t]\}.
\end{aligned} \tag{10}
$$

For $a \neq 3$ (any gauge but Yennie's), it is probably safe to drop the e^4 term and obtain an explicit solution of the integral equation (9). Adopting dimensionless variables

$$\xi = (a-3)e^2/16\pi^2, \qquad z = s/m^2, \qquad \tau(z) = s\sigma(s) \tag{11}$$

the equation simplifies to

$$(z-1)\tau(z) = \xi(z+1) + \xi \int_1^z dz'\,\tau(z')(1+z/z')$$

and this can be cast into the differential form,

$$z(1-z)\tau''(z) + 2z(\xi-1)\tau'(z) + \xi\tau(z) = 0. \tag{12}$$

As in the spinor case, we recognise a hypergeometric equation and the appropriate solution satisfying the boundary conditions,

$$\tau(z) = \frac{2\xi}{z-1}\left(\frac{z-1}{\mu^2/m^2}\right)^{2\xi} F(a, b; c; 1-z),$$

with

$$\begin{aligned} a &= \tfrac{1}{2}[-1+2\xi-(1+4\xi^2)^{1/2}] \\ b &= \tfrac{1}{2}[-1+2\xi+(1+4\xi^2)^{1/2}] \\ c &= 2\xi \end{aligned} \tag{13}$$

incorporates an infrared cut-off (necessary in higher orders of the perturbation series in ξ). Hence we obtain

$$\sigma(W^2) = \frac{2\xi}{W^2-m^2}\left(\frac{W^2-m^2}{\mu^2}\right)^{2\xi} F\left(a+1, b+1; c; 1-\frac{W^2}{m^2}\right)\theta(W^2-m^2). \tag{14}$$

Since we have dropped the order e^4 contribution to $\operatorname{Im}\Sigma$ in deriving (14), it is quite proper to make the approximation

$$a \approx -1+\xi, \qquad b \approx \xi$$

in the hypergeometric function before comparing this with the spinor case:

$$\begin{aligned} \sigma(W) = \epsilon(W)\theta(W^2-m^2)&\frac{2\xi m^2}{W(W^2-m^2)}\left(\frac{W^2-m^2}{\mu^2}\right)^{2\xi} \\ &\times\left[F\left(\xi, \xi; 2\xi; 1-\frac{W^2}{m^2}\right) + \frac{W}{m}F\left(\xi, \xi+1; 2\xi; 1-\frac{W^2}{m^2}\right)\right] \end{aligned} \tag{14'}$$

in Landau gauge. At this level the analogy is perfect. Using (14), we can go on to determine the meson propagator itself

$$\Delta(p) = \frac{1}{p^2-m^2} - \frac{1}{m^2}\left(\frac{m^2}{\mu^2}\right)^{2\xi}\Gamma(1-\xi)\Gamma(2-\xi)\Gamma(1+2\xi)F\left(1-\xi, 2-\xi; 2; \frac{p^2}{m^2}\right) \tag{15}$$

and from the asymptotic expansion as $p^2/m^2 \to \infty$

$$F\left(1-\xi, 2-\xi; 2; \frac{p^2}{m^2}\right)$$
$$= \frac{(-p^2/m^2)^{-1+\xi}}{\Gamma(2-\xi)\Gamma(1+\xi)}\left(1+\frac{m^2}{p^2}\xi(1-\xi)[\ln(-p^2/m^2)+\psi(2)+\psi(1)\right.$$
$$\left. -\psi(2-\xi)-\psi(1+\xi)]+O(p^{-4})\right)$$

and the identifications

$$\Delta(p) \sim \frac{Z_\Phi^{-1}(p^2)}{p^2}\left(1+\frac{m_0^2(p^2)}{p^2}+\ldots\right)$$
$$Z_\Phi^{-1} \equiv \lim_{p^2\to\infty} Z_\Phi^{-1}(p^2), \text{ etc,}$$

one can perceive that Z_Φ^{-1} and m_0^2/m^2 are logarithmically infinite in the zeroth gauge approximation for $\xi > 0$.

What of the e^4 term in Im Σ which distinguishes scalar from spinor electrodynamics and which certainly modifies answer (15)? We have unfortunately been unable to solve (9) in complete generality with it included—the problem looks quite intractable at present for arbitrary a values. In the Yennie gauge ($a = 3$), there is a better chance of an answer since the e^4 term is the *only* component of the absorptive part:

$$\operatorname{Im}\Sigma(p^2, m^2) = c[p^4 - m^4 - 2m^2p^2\ln(p^2/m^2)]; \qquad c \propto e^4.$$

In terms of the previous variables, the equation for $\tau(z) = s\sigma(s)$ then reads

$$(z-1)\tau(z) = \frac{c(z^2-1-2z\ln z)}{z-1} + c\int_1^z dz'\,\frac{\tau(z')}{z'}\,\frac{[z^2-z'^2-2zz'\ln(z/z')]}{z-z'}. \tag{16}$$

The best we can do here is to solve (16) in various limits:

(i) In the infrared region $s \to m^2$ or $z \to 1$, we have

$$(z-1)\tau(z) \approx \tfrac{1}{3}c(z-1)^2 + c\int_1^z dz'(z'-z)^2\tau(z')$$

giving

$$\tau(z) \approx \tfrac{1}{3}c(z-1) + O(z-1)^2.$$

(ii) In the ultraviolet region, $s/m^2 \gg 1$ or $z \to \infty$, we can approximate equation (16) by

$$\tau(z) \approx c\left(1-\frac{2\ln z}{z}\right) + \int_{z^{-1}}^{1} dy\left(1+\frac{1}{y}+\frac{2\ln y}{1-y}\right)\tau(yz)c.$$

Using the mean value theorem, we verify that $\tau(z) \to \text{constant} \propto e^4$ in this limit.

Having exposed the similarity and difference between the gauge covariant solutions of spin-0 and spin-$\frac{1}{2}$ electrodynamics, it will be very interesting next to consider analogous gauge covariant solutions for non-Abelian groups. There the fictitious particles have spinor-like equations while the gauge fields have scalar-like equations and the gauge identities intermix gauge and fictitious particle Green functions. We expect that asymptotic freedom will play an important role in that investigation.

Appendix

This is just to spell out the main points in evaluating the phase space integrals and more particularly some details connected with the absorptive part of the photon propagator:

$$\lim_{\mu^2\to 0}\{-\eta^{\mu\nu}\delta(k^2)+k^\mu k^\nu[\delta(k^2-\mu^2)-\delta(k^2)](1-a)/\mu^2\}$$
$$=[-\eta^{\mu\nu}\delta(k^2)-(1-a)k^\mu k^\nu\delta'(k^2)].$$

The two-body integral

$$-\frac{\mathrm{i}}{\pi}\mathrm{Im}\Big(\int \bar{\mathrm{d}}^4k\, f(2pk, k^2, p^2)/[(p-k)^2-m^2](k^2-\mu^2)\Big)$$
$$=(2\pi)^{-3}\int \mathrm{d}^4k\, f\,\delta_+[(p-k)^2-m^2]\delta_+(k^2-\mu^2)$$
$$=(16\pi^2p^2)^{-1}f(p^2-m^2+\mu^2,\mu^2,p^2)\{[p^2-(m+\mu)^2][p^2-(m-\mu)^2]\}^{1/2}$$

is so familiar as to require no explanation. Derivatives of this with respect to μ^2 can be used to work out the e^2 and some of the e^4 pieces in $\mathrm{Im}\,\Sigma$.

The three-body integrals have to be treated somewhat differently because differentiation of a distribution and the Dalitz region with respect to photon masses is a difficult job. Instead, we proceed as follows: in

$$R=(2\pi)^{-5}\int \mathrm{d}^4k\; \mathrm{d}^4k'\;\delta_+[(p-k-k')^2-m^2]\delta_+(k^2)\delta_+(k'^2)$$

we change variables to

$$K=k+k', \qquad q=\tfrac{1}{2}(k-k')$$

and use the identity

$$\delta(a)\delta(b)=2\delta(a+b)\delta(a-b).$$

Also we can choose frames such that

$$p_\mu=(p;0,0,0)$$
$$K_\mu=(K\cosh\chi;0,0,K\sinh\chi)$$

whereupon R reduces to the integral

$$R=\frac{1}{64\pi^3p}\int \mathrm{d}K^2\,\mathrm{d}q^2\,\mathrm{d}q_0\,\delta(\tfrac{1}{4}K^2+q^2)$$

subject to $0<K<p-m$ and

$$q_0^2\leqslant -q^2\sinh^2\chi=[(p-m)^2-K^2][(p+m)^2-K^2]/16p^2.$$

Hence

$$R=\frac{1}{128\pi^3p^2}\int_0^{(p-m)^2}\mathrm{d}K^2\{[(p-m)^2-K^2][(p+m)^2-K^2]\}^{1/2}=\frac{1}{128\pi^3p^2}\iint \mathrm{d}t\,\mathrm{d}u$$

where $t=(p-k)^2$ and $u=(p-k')^2$ are the Mandelstam variables in the decay region bounded by $tu=m^2p^2$ and $t+u=m^2+p^2$.

In this fashion we can treat the trickier problem of finding

$$R'' = (2\pi)^{-5} \int d^4k\, d^4k'\, \delta_+[(p-k-k')^2 - m^2](k\,.\,k')^2 \delta'_+(k^2)\delta'_+(k'^2).$$

Changing variables as before and putting

$$\delta'(a)\delta'(b) = 2[\delta''(a+b)\delta(a-b) - \delta''(a-b)\delta(a+b)]$$

the integral boils down to

$$R'' = \frac{1}{256\pi^3 p} \int dK^2\, dq^2\, dq_0 (\tfrac{1}{4}K^2 - q^2)^2 \delta''(\tfrac{1}{4}K^2 + q^2)$$

$$= \frac{1}{128\pi^3 p^2} \int dK\{[(p-m)^2 - K^2][(p+m)^2 - K^2]\}^{1/2}(-q^2)^{1/2}$$

$$\times (\tfrac{1}{4}K^2 - q^2)^2 \delta''(\tfrac{1}{4}K^2 + q^2)\, dq^2$$

$$= \frac{5}{512\pi^3 p^2} \int dK^2 \{\ \}^{1/2} = \frac{5}{512\pi^3 p^2} \int\int dt\, du.$$

This makes the origin of the various terms in (10) more comprehensible.

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